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LETTER TO THE EDITOR

Integral expressions for incremental work in electromagnetism

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Abstract. Integral expressions are presented for $\mathrm{d}W$, the incremental work done when a polarisable and magnetisable body is subjected to static electrical and magnetic fields. These exhibit a decomposition into a vacuum term and a contribution of the body to the total value of $\mathrm{d}W$. Existing expressions for the latter are shown to be incorrect and an explanation for this is advanced. Finally, the practical advantages of one particular integral expression for $\mathrm{d}W$ are enumerated.

The most obvious way of defining vacuum terms, for static electrical and magnetic fields, is to say that the work, W, done in increasing the polarisation, $\boldsymbol{P}$, and magnetisation, $\boldsymbol{I}$, of a body contains vacuum terms $\int_\infty (E^2/8\pi)\,\mathrm{d}V$ and $\int_\infty (H^2/8\pi)\,\mathrm{d}V$. Thus, Stoner (1935, 1937) starts from the Maxwell expression $\int_\infty (\boldsymbol{H}\cdot \mathrm{d}\boldsymbol{B}/4\pi)\,\mathrm{d}V$ and, following an earlier treatment of Debye (1925), separates the incremental work, $\mathrm{d}W$, into two components which are, when electrical terms are also included, given by the right-hand side of

$$\frac{1}{4\pi}\int_\infty [\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{D} + \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{B}]\,\mathrm{d}V = \frac{1}{8\pi}\int_\infty [\mathrm{d}(E^2 + H^2)]\,\mathrm{d}V + \int_\infty [\boldsymbol{E}\cdot \mathrm{d}\boldsymbol{P} + \boldsymbol{H}\cdot \mathrm{d}\boldsymbol{I}]\,\mathrm{d}V \qquad (1)$$

where $\boldsymbol{D} = \boldsymbol{E} + 4\pi\boldsymbol{P}$ and $\boldsymbol{B} = \boldsymbol{H} + 4\pi\boldsymbol{I}$. This is a procedure that has been followed ever since (Rhodes 1949, Kittel 1956, Morrish 1965, Nye 1985) even though its shortcomings were pointed out at the time of Stoner's original publications. Guggenheim (1936) commented that there were many different ways of separating the left-hand side of equation (1) into two components, and, indeed, the propriety of regarding the integral involving $E^2 + H^2$ as a vacuum term cannot be sustained when it is recalled that $\boldsymbol{E}$ and $\boldsymbol{H}$ are not the externally applied fields but are the fields as modified by the presence of polarisable and magnetisable material. That there is a difference between these two is well known (Stratton 1941) in connection with the calculation of depolarising and demagnetising factors but it is sufficient, for the moment, to denote by $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ the fields that would be present in all space if the polarisable and magnetisable material were absent (a more precise definition of which is given in the following two paragraphs). What is now needed, therefore, is a vacuum term that depends upon $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ rather than upon $\boldsymbol{E}$ and $\boldsymbol{H}$ (i.e. the fields of electromagnetic theory).

The introduction of two new vectors ($\boldsymbol{E}^0$ and $\boldsymbol{H}^0$) into $\mathrm{d}W$ can only be accomplished by adding to the integrand of (1) terms that vanish when integrated over all space (so leaving the value of $\mathrm{d}W$ unaltered). In principle, there is an infinite number of ways

(O'Rahilly 1938) in which this can be done but many of these will involve the construction of an integral that vanishes only by virtue of a presumed degree of symmetry in the geometry of the arrangements for producing the electromagnetic fields and/or the disposition of polarisable material. A more universal solution to the problem is provided by a theorem enunciated by Stratton (1941) which states that the integral over all space of the scalar product of an irrotational vector $\boldsymbol{Q}$ and a solenoidal vector $\boldsymbol{R}$ is zero, provided that $\boldsymbol{Q}$ and $\boldsymbol{R}$ and their derivatives are continuous everywhere except on a finite number of closed surfaces, across which it is required only that the tangential component of $\boldsymbol{Q}$ and the normal component of $\boldsymbol{R}$ both be continuous. This theorem allows the left-hand side of equation (1) to be transformed into expressions that involve the fields $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$.

For a single polarisable and magnetisable body, of volume V, that is subjected to electrical and magnetic fields provided by charge and current sources (of densities ρ and $\boldsymbol{J}$, respectively) in free space, $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ can be defined as the fields produced by these same charge and current sources when the body itself is absent. Thus

$$\operatorname{div} \boldsymbol{D} = \rho = \operatorname{div} \boldsymbol{D}^0 \tag{2a}$$

$$\operatorname{curl} \boldsymbol{H} = 4\pi \boldsymbol{J}/c = \operatorname{curl} \boldsymbol{H}^0 \tag{2b}$$

$$\boldsymbol{E}^0 = \boldsymbol{D}^0 \tag{2c}$$

$$\boldsymbol{H}^0 = \boldsymbol{B}^0 \tag{2d}$$

These equations now allow the theorem of Stratton (1941) to be used to express the left-hand side of equation (1) in a number of different forms; if, additionally, $\mathrm{d}[(E^0)^2 + (H^0)^2]$ is allowed to appear in the integrand with no multiplicative factor other than unity, then there remain a number of equivalent forms, one of which is

$$\mathrm{d}W = \frac{1}{8\pi}\int_\infty \mathrm{d}[(E^0)^2 + (H^0)^2]\,\mathrm{d}V + \frac{1}{4\pi}\int_\infty [(\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{D}^0 - \boldsymbol{E}^0\cdot\mathrm{d}\boldsymbol{D}) + (\boldsymbol{H}^0\cdot\mathrm{d}\boldsymbol{B} - \boldsymbol{H}\cdot\mathrm{d}\boldsymbol{B}^0)]\,\mathrm{d}V \tag{3}$$

and another of which is

$$\mathrm{d}W = \frac{1}{8\pi}\int_\infty \mathrm{d}[(E^0)^2 + (H^0)^2]\,\mathrm{d}V + \frac{1}{4\pi}\int_\infty [(\boldsymbol{E}\cdot\mathrm{d}\boldsymbol{D}^0 - \boldsymbol{D}\cdot\mathrm{d}\boldsymbol{E}^0) + (\boldsymbol{H}^0\cdot\mathrm{d}\boldsymbol{B} - \boldsymbol{B}^0\cdot\mathrm{d}\boldsymbol{H})]\,\mathrm{d}V = \frac{1}{8\pi}\int_\infty \mathrm{d}[(E^0)^2 + (H^0)^2]\,\mathrm{d}V + \int_V [\boldsymbol{H}^0\cdot\mathrm{d}\boldsymbol{I} - \boldsymbol{P}\cdot\mathrm{d}\boldsymbol{E}^0]\,\mathrm{d}V. \tag{4}$$

All these equivalent forms are now universal in the sense that they hold for all field patterns and do not rely on any presumed symmetry of geometrical arrangements.

In the existing literature that (correctly) expresses the vacuum term as a function of $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ (rather than $\boldsymbol{E}$ and $\boldsymbol{H}$), the form of $\mathrm{d}W$ that is most commonly assumed is

$$\mathrm{d}W = \frac{1}{8\pi}\int_\infty \mathrm{d}[(E^0)^2 + (H^0)^2]\,\mathrm{d}V + \int_V [\boldsymbol{H}^0\cdot\mathrm{d}\boldsymbol{I} + \boldsymbol{E}^0\cdot\mathrm{d}\boldsymbol{P}]\,\mathrm{d}V. \tag{5}$$

This is the form quoted in textbooks that consider the thermodynamics of polarisable and magnetisable materials (Pippard 1957, Callen 1960), where the magnetic case is treated in some detail but the electrical case is only dealt with by analogy. It is the form assumed in investigations (Ascher 1975, Schmid 1975) of the magnetoelectric properties of crystalline materials and, more generally, in the Landau–Ginsburg–Devonshire formalism for ferroic† materials as set out by Newnham (1974) and by Newnham and Cross (1976). However, it is clear that equation (5) does not agree with equation (4)—and therefore (3)—since the difference between (5) and (4) is $\int_V[\mathrm{d}(\boldsymbol{E}^0 \cdot \boldsymbol{P})]\, \mathrm{d}V$, which certainly does not vanish.

The explanation for this is that, however attractive it may seem, it is not actually legitimate to treat the electrical case by analogy with the magnetic (or vice versa) because of an essential difference between the field patterns of $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$. To illustrate this it is convenient to consider the case in which $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ are uniform and parallel to each other within V. As field conditions are investigated along a line perpendicular to this common direction, then at first (i.e. within V) $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ are parallel but eventually, at large distances, $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ must become antiparallel, because of the solenoidal nature of $\boldsymbol{E}^0$ and the irrotational nature of $\boldsymbol{H}^0$. However large the extent of the region in which $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ are uniform (and parallel) is made, eventually this lack of symmetry between the two field patterns will manifest itself and the two are therefore not truly analogous. (Although less obvious in its consequences, this lack of symmetry is, of course, also evident more generally in the Maxwell equations, $\operatorname{div} \boldsymbol{E}^0 = \rho$ and $c \operatorname{curl} \boldsymbol{H}^0 = 4\pi \boldsymbol{J}$, that connect E^0 and H^0 with their sources.)

It may therefore be concluded that equation (5), although very widely accepted, is actually incorrect and that the correct value for $\mathrm{d}W$ is given by equation (3), equation (4) or one of the expressions equivalent to them. All of these correctly separate off the integral of $\mathrm{d}[(E^0)^2 + (H^0)^2]$ over all space as a vacuum term which represents the contribution to $\mathrm{d}W$ that exists in the absence of the body; this is a quantity that is independent of the thermodynamic state and properties of the body. The remainder of $\mathrm{d}W$ is therefore the contribution of the body to the total value of $\mathrm{d}W$, or, more exactly, the difference in incremental work which corresponds to the modification of the electromagnetic fields from those appropriate to free space (i.e. $\boldsymbol{E}^0, \boldsymbol{D}^0 \equiv \boldsymbol{E}^0, \boldsymbol{H}^0, \boldsymbol{B}^0 \equiv \boldsymbol{H}^0$) to those obtaining when the body is present (i.e. $\boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}$).

As between the various correct expressions for $\mathrm{d}W$ given by equation (3), equation (4) and the other expressions equivalent to them, equation (4) stands out as being the most convenient for practical purposes. First, it is convenient in practice that $\mathrm{d}W$ is expressed not in terms of the internal fields $\boldsymbol{E}$ and $\boldsymbol{H}$ but rather in terms of the external fields $\boldsymbol{E}^0$ and $\boldsymbol{H}^0$ applied to the body. Secondly, the contribution of the body to the total value of $\mathrm{d}W$ has the form of an integral over V in equation (4), so that, in calculating $\mathrm{d}W$, it is not necessary to know how the presence of the body modifies the electromagnetic fields outside the body. Finally, it is convenient in practice that this contribution is expressed in terms of the polarisation and magnetisation of the body itself, because there exist shapes of body for which it is easy to ensure that $\boldsymbol{P}$ and $\boldsymbol{I}$ are uniform throughout V: for example, if the body is ellipsoidal in shape, then uniform polarisation and magnetisation are produced (Stratton 1941) by placing it in fields that were originally uniform. In view of the practical importance of applying uniform fields to ellipsoidal or,

† The term 'ferroic' was introduced by Aizu (1972) to describe a crystal (i) that has two or more orientation states in the absence of magnetic, electric and mechanical external forces and (ii) that can go from one to another of these orientation states by the application of some of such forces. Any two of the orientation states are identical or enantiomorphous in crystal structure, but differ with respect to direction of arrangement of the atoms which may bear an electric charge, an electric dipole moment and a magnetic dipole moment.

more commonly, spherical specimens, the use of equation (4) is therefore extremely convenient in evaluating dW.

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